1.2 MATHEMATICAL INDUCTION

If a flight of stairs is designed so that falling off any step inevitably leads to falling off the next, then falling off the first step is a sure way to end up at the bottom. Crudely expressed, this is the essence of the *principle of mathematical induction*: If the truth of a statement depending on a given integer n implies the truth of the corresponding statement with n replaced by $n + 1$, then the statement is true for all positive integers n if it is true for $n = 1$. Although you have probably studied this principle before, it is so important that it merits careful review here.

Peano's Postulates and Induction

The rigorous construction of the real number system starts with a set $\mathbb{N}$ of undefined elements called *natural numbers*, with the following properties.

(A) $\mathbb{N}$ is nonempty.

(B) Associated with each natural number n there is a unique natural number n' called the *successor of* n.

(C) There is a natural number $\overline{n}$ that is not the successor of any natural number.

(D) Distinct natural numbers have distinct successors; that is, if $n \neq m$, then $n' \neq m'$.

(E) The only subset of $\mathbb{N}$ that contains $\overline{n}$ and the successors of all its elements is $\mathbb{N}$ itself.

These axioms are known as *Peano's postulates*. The real numbers can be constructed from the natural numbers by definitions and arguments based on them. This is a formidable task that we will not undertake. We mention it to show how little you need to start with to construct the reals and, more important, to draw attention to postulate **(E)**, which is the basis for the principle of mathematical induction.

It can be shown that the positive integers form a subset of the reals that satisfies Peano's postulates (with $\overline{n} = 1$ and $n' = n + 1$), and it is customary to regard the positive integers and the natural numbers as identical. From this point of view, the principle of mathematical induction is basically a restatement of postulate **(E)**.

Theorem 1.2.1 (Principle of Mathematical Induction) *Let* P_1, P_2,..., P_n,
... *be propositions, one for each positive integer, such that*

(a) P_1 *is true*;

(b) *for each positive integer* n, P_n *implies* P_{n+1}.

Then P_n *is true for each positive integer* n.

Proof Let

$$\mathbb{M} = \left\{ n \,\middle|\, n \in \mathbb{N} \text{ and } P_n \text{ is true} \right\}.$$

From **(a)**, $1 \in \mathbb{M}$, and from **(b)**, $n + 1 \in \mathbb{M}$ whenever $n \in \mathbb{M}$. Therefore, $\mathbb{M} = \mathbb{N}$, by postulate **(E)**. ☐

Example 1.2.1 Let P_n be the proposition that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

Then P_1 is the proposition that $1 = 1$, which is certainly true. If P_n is true, then adding $n + 1$ to both sides of (1) yields

$$\begin{aligned} (1 + 2 + \cdots + n) + (n + 1) &= \frac{n(n+1)}{2} + (n + 1) \\ &= (n + 1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

or

$$1 + 2 + \cdots + (n + 1) = \frac{(n+1)(n+2)}{2},$$

which is P_{n+1}, since it has the form of (1), with n replaced by $n+1$. Hence, P_n implies P_{n+1}, so (1) is true for all n, by Theorem 1.2.1. ■

A proof based on Theorem 1.2.1 is an *induction proof*, or *proof by induction*. The assumption that P_n is true is the *induction assumption*. (Theorem 1.2.3 permits a kind of induction proof in which the induction assumption takes a different form.)

Induction, by definition, can be used only to verify results conjectured by other means. Thus, in Example 1.2.1 we did not use induction to *find* the sum

$$s_n = 1 + 2 + \cdots + n; \tag{2}$$

rather, we *verified* that

$$s_n = \frac{n(n+1)}{2}. \tag{3}$$

How you guess what to prove by induction depends on the problem and your approach to it. For example, (3) might be conjectured after observing that

$$s_1 = 1 = \frac{1 \cdot 2}{2}, \quad s_2 = 3 = \frac{2 \cdot 3}{2}, \quad s_3 = 6 = \frac{4 \cdot 3}{2}.$$

However, this requires sufficient insight to recognize that these results are of the form (3) for $n = 1, 2$, and 3. Although it is easy to prove (3) by induction once it has been conjectured, induction is not the most efficient way to find s_n, which can be obtained quickly by rewriting (2) as

$$s_n = n + (n-1) + \cdots + 1$$

and adding this to (2) to obtain

$$2s_n = [n+1] + [(n-1)+2] + \cdots + [1+n].$$

There are n bracketed expressions on the right, and the terms in each add up to $n+1$; hence,

$$2s_n = n(n+1),$$

which yields (3).

The next two examples deal with problems for which induction is a natural and efficient method of solution.

Example 1.2.2 Let $a_1 = 1$ and

$$a_{n+1} = \frac{1}{n+1} a_n, \quad n \ge 1 \tag{4}$$

(we say that a_n is defined *inductively*), and suppose that we wish to find an explicit formula for a_n. By considering $n = 1, 2$, and 3, we find that

$$a_1 = \frac{1}{1}, \quad a_2 = \frac{1}{1 \cdot 2}, \quad \text{and} \quad a_3 = \frac{1}{1 \cdot 2 \cdot 3},$$

and therefore we conjecture that

$$a_n = \frac{1}{n!}. \tag{5}$$

This is given for $n = 1$. If we assume it is true for some n, substituting it into (4) yields

$$a_{n+1} = \frac{1}{n+1}\frac{1}{n!} = \frac{1}{(n+1)!},$$

which is (5) with n replaced by $n + 1$. Therefore, (5) is true for every positive integer n, by Theorem 1.2.1. ■

Example 1.2.3 For each nonnegative integer n, let x_n be a real number and suppose that

$$|x_{n+1} - x_n| \le r|x_n - x_{n-1}|, \quad n \ge 1, \tag{6}$$

where r is a fixed positive number. By considering (6) for $n = 1$, 2, and 3, we find that

$$\begin{aligned} &|x_2 - x_1| \le r|x_1 - x_0|, \\ &|x_3 - x_2| \le r|x_2 - x_1| \le r^2|x_1 - x_0|, \\ &|x_4 - x_3| \le r|x_3 - x_2| \le r^3|x_1 - x_0|. \end{aligned}$$

Therefore, we conjecture that

$$|x_n - x_{n-1}| \le r^{n-1}|x_1 - x_0| \quad \text{if} \quad n \ge 1. \tag{7}$$

This is trivial for $n = 1$. If it is true for some n, then (6) and (7) imply that

$$|x_{n+1} - x_n| \le r(r^{n-1}|x_1 - x_0|), \quad \text{so} \quad |x_{n+1} - x_n| \le r^n|x_1 - x_0|,$$

which is proposition (7) with n replaced by $n + 1$. Hence, (7) is true for every positive integer n, by Theorem 1.2.1. ■

The major effort in an induction proof (after $P_1, P_2, \dots, P_n, \dots$ have been formulated) is usually directed toward showing that P_n implies P_{n+1}. However, it is important to verify P_1, since P_n may imply P_{n+1} even if some or all of the propositions $P_1, P_2, \dots, P_n, \dots$ are false.

Example 1.2.4 Let P_n be the proposition that $2n - 1$ is divisible by 2. If P_n is true then P_{n+1} is also, since

$$2n + 1 = (2n - 1) + 2.$$

However, we cannot conclude that P_n is true for $n \ge 1$. In fact, P_n is false for every n. ■

The following formulation of the principle of mathematical induction permits us to start induction proofs with an arbitrary integer, rather than 1, as required in Theorem 1.2.1.

Theorem 1.2.2 *Let n_0 be any integer (positive, negative, or zero). Let P_{n_0}, P_{n_0+1}, $\dots$, P_n, $\dots$ be propositions, one for each integer $n \ge n_0$, such that*

(a) *P_{n_0} is true;*

(b) *for each integer $n \ge n_0$, P_n implies P_{n+1}.*

Then P_n is true for every integer $n \ge n_0$.

Proof For $m \ge 1$, let Q_m be the proposition defined by $Q_m = P_{m+n_0-1}$. Then $Q_1 = P_{n_0}$ is true by **(a)**. If $m \ge 1$ and $Q_m = P_{m+n_0-1}$ is true, then $Q_{m+1} = P_{m+n_0}$ is true by **(b)** with n replaced by $m + n_0 - 1$. Therefore, Q_m is true for all $m \ge 1$ by Theorem 1.2.1 with P replaced by Q and n replaced by m. This is equivalent to the statement that P_n is true for all $n \ge n_0$. □

Example 1.2.5 Consider the proposition P_n that

$$3n + 16 > 0.$$

If P_n is true, then so is P_{n+1}, since

$$\begin{aligned} 3(n+1) + 16 &= 3n + 3 + 16 \\ &= (3n + 16) + 3 > 0 + 3 \text{ (by the induction assumption)} \\ &> 0. \end{aligned}$$

The smallest n_0 for which P_{n_0} is true is $n_0 = -5$. Hence, P_n is true for $n \ge -5$, by Theorem 1.2.2. ■

Example 1.2.6 Let P_n be the proposition that

$$n! - 3^n > 0.$$

If P_n is true, then

$$\begin{aligned} (n+1)! - 3^{n+1} &= n!(n+1) - 3^{n+1} \\ &> 3^n(n+1) - 3^{n+1} \quad \text{(by the induction assumption)} \\ &= 3^n(n-2). \end{aligned}$$

Therefore, P_n implies P_{n+1} if $n > 2$. By trial and error, $n_0 = 7$ is the smallest integer such that P_{n_0} is true; hence, P_n is true for $n \ge 7$, by Theorem 1.2.2. ■

The next theorem is a useful consequence of the principle of mathematical induction.

Theorem 1.2.3 *Let n_0 be any integer (positive, negative, or zero). Let P_{n_0}, $P_{n_0+1}, \dots$, P_n, $\dots$ be propositions, one for each integer $n \ge n_0$, such that*

(a) *P_{n_0} is true;*

(b) *for $n \ge n_0$, P_{n+1} is true if P_{n_0}, $P_{n_0+1}, \dots$, P_n are all true.*

Then P_n is true for $n \ge n_0$.

Proof For $n \geq n_0$, let Q_n be the proposition that $P_{n_0}, P_{n_0+1}, \dots, P_n$ are all true. Then Q_{n_0} is true by **(a)**. Since Q_n implies P_{n+1} by **(b)**, and Q_{n+1} is true if Q_n and P_n are both true, Theorem 1.2.2 implies that Q_n is true for all $n \geq n_0$. Therefore, P_n is true for all $n \geq n_0$. □

Example 1.2.7 An integer $p > 1$ is a *prime* if it cannot be factored as $p = rs$ where r and s are integers and $1 < r, s < p$. Thus, 2, 3, 5, 7, and 11 are primes, and, although 4, 6, 8, 9, and 10 are not, they are products of primes:

$$4 = 2 \cdot 2, \quad 6 = 2 \cdot 3, \quad 8 = 2 \cdot 2 \cdot 2, \quad 9 = 3 \cdot 3, \quad 10 = 2 \cdot 5.$$

These observations suggest that each integer $n \geq 2$ is a prime or a product of primes. Let this proposition be P_n. Then P_2 is true, but neither Theorem 1.2.1 nor Theorem 1.2.2 apply, since P_n does not imply P_{n+1} in any obvious way. (For example, it is not evident from $24 = 2 \cdot 2 \cdot 2 \cdot 3$ that 25 is a product of primes.) However, Theorem 1.2.3 yields the stated result, as follows. Suppose that $n \geq 2$ and $P_2, \dots, P_n$ are true. Either $n + 1$ is a prime or

$$n + 1 = rs, \tag{8}$$

where r and s are integers and $1 < r, s < n$, so P_r and P_s are true by assumption. Hence, r and s are primes or products of primes and (8) implies that $n + 1$ is a product of primes. We have now proved P_{n+1} (that $n + 1$ is a prime or a product of primes). Therefore, P_n is true for all $n \geq 2$, by Theorem 1.2.3. ■